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ON THE TWO q-ANALOGUE LOGARITHMIC FUNCTIONS: $\ln_q(w)$, $\ln\{e_q(z)\}$

Charles A. Nelson[1] and Michael G. Gartley[2]

Department of Physics, State University of New York at Binghamton

Binghamton, N.Y. 13902-6016

Abstract

There is a simple, multi-sheet Riemann surface associated with $e_q(z)$'s inverse function $ln_q(w)$ for $0 < q \leq 1$. A principal sheet for $ln_q(w)$ can be defined. However, the topology of the Riemann surface for $ln_q(w)$ changes each time q increases above the collision point q^*_τ of a pair of the turning points τ_i of $e_q(x)$. There is also a power series representation for $ln_q(1+w)$. An infinite-product representation for $e_q(z)$ is used to obtain the ordinary natural logarithm $\ln\{e_q(z)\}$ and the values of the sum rules $\sigma^e_n \equiv \sum_{i=1}^{\infty} \left(\frac{1}{z_i}\right)^n$ for the zeros z_i of $e_q(z)$. For $|z| < |z_1|$, $e_q(z) = exp\{b(z)\}$ where $b(z) = -\sum_{n=1}^{\infty} \frac{1}{n}\sigma^e_n z^n$. The values of the sum rules for the q-trigonometric functions, σ^c_{2n} and σ^s_{2n+1}, are q-deformations of the usual Bernoulli numbers.

[1]Electronic address: cnelson @ bingvmb.cc.binghamton.edu .

[2]Present address: Rochester Institute of Technology, 1 Lomb Memorial Drive, Rochester, N.Y. 14623.

1 Introduction:

The ordinary exponential and logarithmic functions find frequent and varied applications in all fields of physics. Recently in the study of quantum algebras, the q-exponential function [1] or mapping $w = e_q(z)$ has reappeared [2-4] from a rather dormant status in mathematical physics. This order-zero entire function can be defined by

$$e_q(z) \equiv \sum_{n=0}^{\infty} \frac{z^n}{[n]!} \tag{1}$$

where

$$[n] = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} \tag{2}$$

The series in Eq.(1) converges uniformly and absolutely for all finite z. Since $[n]$ is invariant under $q \to 1/q$, for real q it suffices to study $0 < q \leq 1$. The q-factorial is defined by $[n]! \equiv [n][n-1]\cdots[1]$, $[0]! \equiv 1$. As $q \to 1$, $e_q(z) \to exp(z)$ the ordinary exponential function.

In [5], we reported some of the remarkable analytic and numerical properties of the infinity of zeros, z_i, of $e_q(x)$ for $x < 0$. In particular, as q increases above the first collision point at $q_z^* \approx 0.14$, these zeros collide in pairs and then move off into the complex z plane, see Fig. 1. They move off as (and remain) a complex conjugate pair $\mu_{A,\bar{A}}$. The turning points of $e_q(z)$, i.e. the zeros of the first derivative $e_q^{'}(z) \equiv de_q(z)/dx$, behave in a similar manner. For instance, at $q_\tau^* \approx 0.25$ the first two turning points, τ_1 and τ_2, collide and move off as a complex conjugate pair $\tau_{A,\bar{A}}$.

In this paper, we first show that there is a simple, multi-sheet Riemann surface associated with $w = e_q(z)$'s inverse function $z = ln_q(w)$. As with the usual $ln(w)$ function, the Riemann surface of $z = ln_q(w)$ defines a single-valued map onto the entire complex z plane. Also, as in the usual case when $q = 1$, a principal sheet for $z = ln_q(w)$ can be defined. However, unlike for the ordinary

$ln(w)$ and $exp(z)$, the topology of the Riemann surface for $ln_q(w)$ changes each time q increases above the collision point q^*_τ of a pair of the turning points τ_i of $e_q(z)$. The turning points of $e_q(z)$ can be used to define square-root branch points of $ln_q(w)$ in the complex w plane, i.e. $b_i = e_q(\tau_i)$.

In Sec. 3, we obtain a power series representation for $ln_q(1+w)$.

In the mathematics and physics literature[3], one also finds the exponential function $E_q(z)$ defined by Jackson[7-8]. It also is given by Eq.(1) but with $[n]$ replaced by $[n]_J$ where

$$[n]_J = q^{(n-1)/2}[n] = \frac{1-q^n}{1-q} \tag{3}$$

For $q > 1$, $E_q(z)$ has simpler properties[4]than $e_q(z)$. We also construct the Riemann surface for its inverse function $Ln_q(w)$. With the substitution $[n] \to [n]_J$, the power series representation for $ln_q(1+w)$ also holds for $Ln_q(1+w)$.

Second, in Sec. 4, we use the infinite-product representation [5] for $e_q(z)$ to (i) obtain the ordinary natural logarithm $\ln\{e_q(z)\}$, and to (ii) evaluate for arbitrary integer $n > 0$ the sum rules

$$\sigma^e_n \equiv \sum_{i=1}^{\infty} \left(\frac{1}{z_i}\right)^n \tag{4}$$

for the zeros z_i of $e_q(z)$. Therefore, for c-number arguments

$$e_q(x)e_q(y) = \exp\{b(x) + b(y)\} \tag{5}$$

where $b(x)$ is defined below in Eq.(20). For $|z| < |z_1|$ the modulus of the first zero,

$$b(z) = -\sum_{n=1}^{\infty} \frac{1}{n}\sigma^e_n z^n \tag{6}$$

[3]Recent reviews of quantum algebras are listed in [6].

[4] For $0 < q < 1$, $E_q(z)$ is a meromorphic function whose power series converges uniformly and absolutely for $|z| < (1-q)^{-1}$ but diverges otherwise. However by the relation, $E_s(x)E_{1/s}(-x) = 1$ for s real, results for $q > 1$ can be used for $0 < q < 1$, see Ref. [5].

We also obtain the logarithms and values of the associated sum rules for all derivatives and integrals of $e_q(x)$, and for the associated q-trigonometric functions [1,5] $cos_q(z)$ and $sin_q(z)$. These results also hold for the analogous functions involving $[n]_J$.

Sec. 5 contains some concluding remarks. In particular, the values of the sum rules for the q-trigonometric functions, σ^c_{2n} and σ^s_{2n+1}, are q-deformations of the usual Bernoulli numbers.

2 Riemann Surfaces of q-Analogue Logarithmic Functions $ln_q(w)$ and $Ln_q(w)$:

For two reasons, we begin by first analyzing the Riemann surface associated with the mapping of Jackson's exponential function $w = u + iv = E_q(z)$ and of its inverse $z = x + iy = Ln_q(w)$. First, the generic structure of the Riemann surface for $Ln_q(w)$ for $q^E > 1$ is the same as that for $ln_q(w)$ for $q^e < (q^* \approx 0.14)$. Second, as q^e varies the topology of the Riemann surface changes for $ln_q(w)$ but the topology remains invariant for $Ln_q(w)$ for all $q^E > 1$. Normally we will suppress the superscripts "E or e" on the q's for there should be no confusion.

2.1 Riemann surface for $Ln_q(w)$:

Figs. 2 and 3 show the Riemann sheet structure and the mappings of Jackson's exponential function $w = E_q(z)$ and of its inverse $z = Ln_q(w)$ for $q^E \approx 1.09$. These figures suffice for illustrating the Riemann sheet for all $q > 1$ because the zeros and turning points of $E_q(z)$ do not collide, but simply move along the negative x axis and out to infinity as $q \to 1$.

These figures also illustrate the Riemann surface for $w = e_q(z)$ and $z = ln_q(w)$ but only prior

to the collision of the first pair of zeros at $q \approx 0.14$.

Notice that the imaginary part $Im\{e_q(z)\} = 0$ on all "solid" contour lines in Fig. 2b whereas the real part $Re\{e_q(z)\} = 0$ on all "dashed" contour lines. The turning points in the complex z plane are denoted by small dark squares, whereas their associated branch points in w are denoted by small dark circles.

Numerically, for $q^E \approx 1.09$, the first 4 zeros of $E_q(z)$ are located at $-12.1111, -13.2011,$ $-14.3892, -15.6842$. The first 4 turning points and $Ln_q(w)$'s branch points (b_i in 10^{-11} units) are respectively at $(\tau_i, b_i) = (-12.4, -43), (-13.6, 5.0), (-14.9, -1.8), (-16.3, 4.4)$. Since $q^E \approx 1$, the asymptotic formula in [5] for τ_i^E is a bad approximation for these values.

Figures for the lower-sheets of a Riemann surface w are omitted in this paper since they simply have the conjugate structures, per the Schwarz reflection principle.

2.2 Riemann surface for $ln_q(w)$:

For $q <\approx 0.14$, Figs. 1-3 also show the topology and branch point structure for the mappings $w = e_q(z)$ and its inverse $z = ln_q(w)$.

Figs. 4-5 are for after the collision of the first pair of zeros of $e_q(z)$ but prior to the collision of the first pair of its turning points, so the structure shown is generic for $0.14 < q < 0.25$. Note that $w_A = e_q(\mu_A) = 0$ occurs as an analytic point for $w = e_q(z)$ which is not possible for the ordinary $exp(z)$ in the finite z plane.

Numerically, Figs. 4-5 are for $q \approx 0.22$; the first 2 zeros of $e_q(z)$ are located at $\mu_A = -2.51 + i0.87, \mu_{\bar{A}} = \bar{\mu_A}$. The first 2 turning points and $ln_q(w)$'s branch points (b_i in 10^{-3} units) are respectively at $(\tau_i, b_i) = (-2.6, 47.70), (-4.7, 69.36)$.

Figs. 6-8 are for after the collision of the first pair of turning points of $e_q(z)$. The topology of the Riemann surface has a new inter-surface structure due to this collision; the figures and their captions explain this new structure. In particular versus Fig. 5, following the collision at $q^*_\tau \approx 0.25$, there no longer exists the $b_1 - b_2$ passage from the lower-half of the principal w sheet to the first lower w sheet. Instead, the b_A passages are to the second upper w sheet.

Numerically, Figs. 6-8 are for $q \approx 0.35$. The first 2 zeros of $e_q(z)$ are now located at $\mu_A = -2.8222 + i1.969$, $\mu_{\bar{A}} = \bar{\mu_A}$; the third zero remains on the negative real axis at $\mu_3 = -5.19755$. The first 4 turning points and $ln_q(w)$'s branch points (b_i in 10^{-3} units) are respectively at $(\tau_i, b_i) = (-3.5434 \pm i1.32945, 22.2415 \pm i18.79), (-6.3471, -9.09587), (-10.7028, 87.536)$. In Figs. 7-8, for clarity of illustration, the position of b_A has been displaced from its true position.

3 Power Series Representations for $\ln_q(1+w)$ and $Ln_q(1+w)$:

To obtain the power series for $\ln_q(1+w)$, we write

$$\begin{aligned} \ln_q(1+w) &= c_1 w + c_2 w^2 + ... \\ &= \sum_{n=1}^{\infty} c_n w^n \end{aligned} \tag{7}$$

Then for $a = \ln_q(1+w)$,

$$\begin{aligned} e_q^a &= 1 + a + \tfrac{a^2}{[2]!} + ... \\ &= 1 + w \end{aligned} \tag{8}$$

So by equating coefficients, we find

$$\begin{gathered} c_1 = 1 \\ c_n = -\sum_{l=2}^{n} \frac{1}{[l]!} \left\{ \sum_{(k_1,k_2,\cdots k_l)} c_{k_1} c_{k_2} \cdots c_{k_l} \right\}, n \geq 2 \end{gathered} \tag{9}$$

In order to follow later expressions in this paper, it is essential to understand the second summation $\sum\limits_{(k_1,k_2,\cdots k_l)}$:

In it, each $k_i =$ "positive integer", $i = 1, 2, \ldots l$.

$(k_1, k_2, \cdots k_l)$ denotes that, for fixed n and l, the summation is the symmetric permutations of each partition of n which satisfy the condition "$k_1 + k_2 + \cdots k_l = n$".

For instance, for $n = 4$:

$$\begin{gathered} \sum_{(k_1,k_2,k_3,k_4)} c_{k_1} c_{k_2} c_{k_3} c_{k_4} = \{c_1 c_1 c_1 c_1\} = (c_1)^4 \\ \sum_{(k_1,k_2,k_3)} c_{k_1} c_{k_2} c_{k_3} = \{c_1 c_1 c_2 + c_1 c_2 c_1 + c_2 c_1 c_1\} = 3 c_1 c_1 c_2 \\ \sum_{(k_1,k_2)} c_{k_1} c_{k_2} = \{c_2 c_2\} + \{c_1 c_3 + c_3 c_1\} = (c_2)^2 + 2 c_1 c_3 \end{gathered} \tag{10}$$

This power series for $\ln_q(1+w)$ is expected to converge only for some w domain, e.g. for $w \leq$ "modulus of distance to the nearest branch point". Note that as $q \to 0$, $w = e_q(z) \to w = 1 + z$ and $z = \ln_q(w) \to z = w - 1$, so $e_q\{\ln_q(w)\} \to e_q\{w - 1\} \to w$.

Thus, the first few terms give

$$\begin{gathered} \ln_q(1+w) = w - \frac{1}{[2]!} w^2 - \left\{ \frac{1}{[3]!} - \frac{2}{[2]![2]!} \right\} w^3 \\ - \left\{ \frac{1}{[4]!} - \frac{2}{[2]!} \left(\frac{1}{[3]!} - \frac{2}{[2]![2]!} \right) + \left(\frac{1}{[2]!} \right)^3 - \frac{3}{[3]![2]!} \right\} w^4 + \ldots \\ = w - \frac{1}{[2]!} w^2 - \left\{ \frac{1}{[3]!} - 2 \left(\frac{1}{[2]!} \right)^2 \right\} w^3 \\ - \left\{ \frac{1}{[4]!} - \frac{5}{[3]![2]!} + 5 \left(\frac{1}{[2]!} \right)^3 \right\} w^4 + \ldots \end{gathered} \tag{11}$$

Notice that here the q-derivative operation defines a new function, $d \ln_q(w)/d_q w \equiv \ln_q(w)' \neq \frac{1}{w}$,

because it does *not* yield a known q-special function since

$$\begin{aligned}\tfrac{d}{d_q w}\ln_q(1+w) &= 1 - w - \left\{\tfrac{1}{[2]!} - 2[3]\left(\tfrac{1}{[2]!}\right)^2\right\} w^2 \\ &- \left\{\tfrac{1}{[3]!} - \tfrac{5[4]}{[3]![2]!} + 5[4]\left(\tfrac{1}{[2]!}\right)^3\right\} w^3 + \cdots\end{aligned} \tag{12}$$

unlike [5] for $e_q(z)$, $\cos_q(z)$, and $\sin_q(z)$.

4 Natural Logarithms and Sum Rules for $e_q(z)$ and Related Functions:

By the Hadamard-Weierstrass theorem, it was shown in Ref.[5] that the following order-zero entire functions have infinite product representations in terms of their respective zeros:

$$e_q(z) = \prod_{i=1}^{\infty}\left(1 - \frac{z}{z_i}\right) \tag{13}$$

$$\begin{aligned} e_q^{(r)}(x) \equiv \tfrac{d^r}{dx^r} e_q(x) &= \alpha_r \prod_{i=1}^{\infty}\left(1 - \tfrac{x}{z_i^{(r)}}\right); r = 1, 2, \ldots \\ \alpha_r &= \tfrac{r!}{[r]!}\end{aligned} \tag{14}$$

$$\begin{aligned} e_q^{(-r)}(x) &= \textstyle\int^x dx_1 \int^{x_1} dx_2 \ldots \int^{x_r} dx_r e_q(x_r) + poly.deg.(r-1), r \geq 1 \\ &\equiv \sum_{n=0}^{\infty} \tfrac{n!}{(n+r)!}\tfrac{x^{n+r}}{[n]!} \\ &= \left(\tfrac{x^r}{r!}\right)\prod_{i=1}^{\infty}\left(1 - \tfrac{x}{z_i^{(-r)}}\right)\end{aligned} \tag{15}$$

$$\begin{aligned}\cos_q(z) &\equiv \sum_{n=0}^{\infty}(-)^n \tfrac{z^{2n}}{[2n]!} \\ &= \prod_{i=1}^{\infty}\left(1 - \left(\tfrac{z}{c_i}\right)^2\right)\end{aligned} \tag{16}$$

$$\begin{aligned}\sin_q(z) &\equiv \sum_{n=0}^{\infty}(-)^n \tfrac{z^{2n+1}}{[2n+1]!} \\ &= z\prod_{i=1}^{\infty}\left(1 - \left(\tfrac{z}{s_i}\right)^2\right)\end{aligned} \tag{17}$$

4.1 Derivation of $\ln\{e_q(z)\}$ and of the values of $\sigma_n^e \equiv \sum\limits_{i=1}^{\infty}\left(\frac{1}{z_i}\right)^n$:

By taking the ordinary natural logarithm of

$$e_q(z) = \prod_{i=1}^{\infty}\left(1 - \frac{z}{z_i}\right), \tag{18}$$

we obtain

$$\begin{aligned} \ln\{e_q(z)\} &= \sum_{i=1}^{\infty} \ln\left\{1 - \tfrac{z}{z_i}\right\} \\ &= -z\left\{\sum_{i=1}^{\infty}\left(\tfrac{1}{z_i}\right)\right\} - \tfrac{z^2}{2}\left\{\sum_{i=1}^{\infty}\left(\tfrac{1}{z_i}\right)^2\right\} - \tfrac{z^3}{3}\left\{\sum_{i=1}^{\infty}\left(\tfrac{1}{z_i}\right)^3\right\}\dots \\ &= b(z) \end{aligned} \tag{19}$$

where the function

$$\begin{aligned} b(z) &\equiv \sum_{i=1}^{\infty} \ln\left\{1 - \tfrac{z}{z_i}\right\} \\ &= -\sum_{n=1}^{\infty} \tfrac{1}{n}\sigma_n^e z^n, \; |z| < |z_1| \end{aligned} \tag{20}$$

Fig. 7 of Ref. [5] shows the polar part $\rho_i = |z_i|$ of the first 8 zeros of $e_q(z)$ for $\approx 0.1 < q <\approx 0.95$. Note that $\rho_i > \rho_{i-1} \geq \rho_1$ where ρ_1 is the modulus of the first zero. The function $b(z) = \ln\{e_q(z)\}$ is thereby expressed in terms of the sum rules for the zeros of $e_q(z)$ since

$$\sigma_n^e \equiv \sum_{i=1}^{\infty}\left(\frac{1}{z_i}\right)^n ; n = 1, 2, \dots \tag{21}$$

By Eq.(20), the multi-sheet Riemann surface of $b(z) = \ln\{e_q(z)\}$ consists of logarithmic branch points at the zeros, z_i, of $e_q(z)$.

The basic properties of $e_q(x)$ displayed in Fig. 1 for $q = 0.1$ follow simply from these expressions for $b(u)$. For instance, the zeros of $e_q(x)$ correspond to where $b(u)$ diverges. A sign change of $e_q(x)$ is due to the principal-value phase change of "$+i\pi$" at the branch point of $\ln\left\{1 - \frac{z}{z_i}\right\}$.

Next, to evaluate these sum rules we proceed as in the above derivation of the power series

representation for $\ln_q(1+w)$. We simply expand both sides of

$$\begin{gathered} e_q(z) = e^{b(z)} \\ 1 + \frac{z}{[1]!} + \frac{z^2}{[2]!} + \ldots = 1 + \frac{b}{1!} + \frac{b^2}{2!} + \ldots \end{gathered} \tag{22}$$

Equating coefficients then gives a recursive formula[5] for these sum rules:

$$\begin{gathered} \sigma_1^e = -1 \\ \sigma_n^e = n \left\{ \sum_{l=2}^{n} \frac{(-)^l}{l!} \left(\sum_{(k_1,k_2,\cdots k_l)} \frac{\sigma_{k_1}\sigma_{k_2}\cdots\sigma_{k_l}}{k_1 k_2 \cdots k_l} \right) - \frac{1}{[n]!} \right\}, n \geq 2 \end{gathered} \tag{23}$$

The notation in the second summation is explained following Eq.(9) for $\ln_q(1+w)$.

The first such sum rules are:

$$\begin{gathered} \sigma_1^e = -1 \\ \sigma_2^e = 1 - \frac{2}{[2]!} \\ \sigma_3^e = -1 + \frac{3}{[2]!} - \frac{3}{[3]!} \\ \sigma_4^e = 1 - \frac{4}{[2]!} + \frac{4}{[3]!} - \frac{4}{[4]!} + \frac{2}{[2]![2]!} \end{gathered} \tag{24}$$

The values of σ_n^e can also be directly obtained from

$$\sigma_n^e = n \sum_{l=1}^{n} \frac{(-)^l}{l} \left\{ \sum_{(k_1,k_2,\cdots k_l)} \frac{1}{[k_1]![k_2]!\cdots[k_l]!} \right\}. \tag{25}$$

Eq.(25) follows by expanding Eq.(19)

$$\begin{gathered} b(z) = -\sum_{n=1}^{\infty} \frac{1}{n} \sigma_n^e z^n = ln(1+y) \\ = y - \frac{y^2}{2} + \frac{y^3}{3} + \ldots \end{gathered} \tag{26}$$

where

$$\begin{gathered} y = e_q(z) - 1 \\ = \frac{z}{[1]!} + \frac{z^2}{[2]!} + \frac{z^3}{[3]!} + \ldots \end{gathered} \tag{27}$$

[5] These σ_n^e sum rules can also be evaluated [5] by expanding both sides of an infinite-product representation of $e_q(z)$. In this way, from σ_n^e for the first few n, we first discovered the general formula Eq.(23) and Eq.(25). Eq.(23) describes a pattern similar to that occurring in the reversion (inversion) of power series.

and then equating coefficients of z^n.

Equivalently, these formulas can be interpreted as representations of the reciprocals of the "bracket" factorials in terms of sums of the reciprocals of the zeros of $e_q(z)$:

$$\begin{aligned} \tfrac{1}{[2]!} &= \tfrac{1}{2!} - \tfrac{1}{2}\sigma_2^e \\ \tfrac{1}{[3]!} &= \tfrac{1}{3!} - \tfrac{1}{2}\sigma_2^e - \tfrac{1}{3}\sigma_3^e \\ \tfrac{1}{[4]!} &= \tfrac{1}{4!} - \tfrac{1}{4}\sigma_2^e - \tfrac{1}{3}\sigma_3^e - \tfrac{1}{4}\sigma_4^e + \tfrac{1}{8}(\sigma_2^e)^2 \end{aligned} \tag{28}$$

The results in this subsection also give $\ln\{E_q(z)\}$ for the analogous $E_q(z)$ for $q > 1$ by the substitution $[n] \to [n]_J$.

4.2 Logarithms and sum rules for related q-analogue functions:

(i) For the "r-*th*" derivative of $e_q(x)$, $e_q^{(r)}(x) \equiv \frac{d^r}{dx^r}e_q(x)$, we similary obtain $[\alpha_r \equiv \frac{r!}{[r]!}]$

$$\begin{aligned} \ln\left\{e_q^{(r)}(x)\right\} &= \ln\alpha_r + b^{(r)}(x); r = 1, 2, \ldots \\ b^{(r)}(z) &= \sum_{i=1}^{\infty} \ln\left(1 - \tfrac{z}{z_i^{(r)}}\right) \end{aligned} \tag{29}$$

where the sum rules for the zeros of the "r-*th*" derivative of $e_q(x)$ are

$$\sigma_n^{(r)} \equiv \sum_{i=1}^{\infty} \left(\frac{1}{z_i^{(r)}}\right)^n. \tag{30}$$

The values of these $e_q(z)$ derivative sum rules are

$$\begin{aligned} \sigma_1^{(r)} &= -\tfrac{r+1}{[r+1]} \\ \sigma_n^{(r)} &= n\left\{\sum_{l=2}^{n} \tfrac{(-)^l}{l!}\left(\sum_{(k_1,k_2,\cdots k_l)} \tfrac{\sigma_{k_1}^{(r)}\sigma_{k_2}^{(r)}\cdots\sigma_{k_l}^{(r)}}{k_1k_2\cdots k_l}\right) - L_n^{(r)}\right\} \end{aligned} \tag{31}$$

where the $L_n^{(r)}$term is given by

$$\begin{aligned} L_n^{(r)} &= \tfrac{(n+r)(n+r-1)\cdots(n+1)}{[n+r]!}\tfrac{1}{\alpha_r} \\ &= \tfrac{(r+n)(r+n-1)\cdots(r+1)}{[r+n][r+n-1]\cdots[r+1]}\tfrac{1}{n!} \end{aligned} \tag{32}$$

Equivalently,

$$\sigma_n^{(r)} = n \sum_{l=1}^{n} \frac{(-)^l}{l} \left\{ \sum_{(k_1,k_2,\cdots k_l)} L_{k_1}^{(r)} L_{k_2}^{(r)} \cdots L_{k_l}^{(r)} \right\} \tag{33}$$

Thus, the "r-th" derivative of $e_q(z)$ is

$$e_q^{(r)}(z) = \frac{r!}{[r]!} \exp\left\{ b^{(r)}(z) \right\} \tag{34}$$

where $b^{(r)}(z) = -\sum\limits_{n=1}^{\infty} \frac{1}{n}\sigma_n^{(r)} z^n$, $|z| < |z_1^{(r)}|$.

(ii) For the "r-th" integral of $e_q(z)$ which is defined in Eq.(15), we obtain $[\beta_r \equiv \frac{1}{r!}]$

$$\begin{gathered} \ln\left\{ \frac{e_q^{(-r)}(x)}{x^r} \right\} = \ln \beta_r + b^{(-r)}(x); r = 1, 2, \ldots \\ b^{(-r)}(z) = \sum_{i=1}^{\infty} \ln\left(1 - \frac{z}{z_i^{(-r)}}\right) \end{gathered} \tag{35}$$

where the associated sum rules are

$$\sigma_n^{(-r)} \equiv \sum_{i=1}^{\infty} \left(\frac{1}{z_i^{(-r)}} \right)^n . \tag{36}$$

The values of these $e_q(z)$ integral sum rules are

$$\begin{gathered} \sigma_1^{(-r)} = -\frac{1}{r+1} \\ \sigma_n^{(-r)} = n \left\{ \sum_{l=2}^{n} \frac{(-)^l}{l!} \left(\sum_{(k_1,k_2,\cdots k_l)} \frac{\sigma_{k_1}^{(-r)} \sigma_{k_2}^{(-r)} \cdots \sigma_{k_l}^{(-r)}}{k_1 k_2 \cdots k_l} \right) - \frac{r!n!}{(r+n)![n]!} \right\} \end{gathered} \tag{37}$$

Equivalently,

$$\sigma_n^{(-r)} = n \sum_{l=1}^{n} \frac{(-)^l}{l} \left\{ \sum_{(k_1,k_2,\cdots k_l)} L_{k_1}^{(-r)} L_{k_2}^{(-r)} \cdots L_{k_l}^{(-r)} \right\} \tag{38}$$

where the $L_m^{(-r)}$ expression

$$L_m^{(-r)} \equiv \frac{r!m!}{(r+m)![m]!} \tag{39}$$

is also the $l = 1$ term in Eq.(37).

Thus, the "r-th" integral of $e_q(z)$ is

$$e_q^{(-r)}(z) = \frac{z^r}{r!} \exp\left\{ b^{(-r)}(z) \right\} \tag{40}$$

where $b^{(-r)}(z) = -\sum_{n=1}^{\infty} \frac{1}{n} \sigma_n^{(-r)} z^n$, $|z| < |z_1^{(-r)}|$.

(iii) For the q-trigonometric functions, we obtain for the $cos_q(z)$ function the representation

$$\begin{aligned} cos_q(z) &= \exp\{b^c(z)\} \\ b^c(z) &= \sum_{i=1}^{\infty} \ln\left(1 - \left(\tfrac{z}{c_i}\right)^2\right) = -\sum_{n=1}^{\infty} \tfrac{1}{n} \sigma_{2n}^c z^{2n}, |z| < |c_1| \end{aligned} \tag{41}$$

where

$$\sigma_{2n}^c \equiv \sum_{i=1}^{\infty} \left(\frac{1}{c_i^2}\right)^n . \tag{42}$$

The values of the cosine sum rules are

$$\begin{aligned} \sigma_2^c &= \sum_{i=1}^{\infty} \left(\tfrac{1}{c_i}\right)^2 = \tfrac{1}{[2]!} \\ \sigma_4^c &= \sum_{i=1}^{\infty} \left(\tfrac{1}{c_i}\right)^4 = \left(\tfrac{1}{[2]!}\right)^2 - \tfrac{2}{[4]!} \\ \sigma_6^c &= \sum_{i=1}^{\infty} \left(\tfrac{1}{c_i}\right)^6 = \left(\tfrac{1}{[2]!}\right)^3 - \tfrac{3}{[2]![4]!} + \tfrac{3}{[6]!} \\ \sigma_{2n}^c &= n\left\{ \sum_{l=2}^{n} \tfrac{(-)^l}{l!} \left(\sum_{(k_1,k_2,\cdots k_l)} \tfrac{\sigma_{2k_1}^c \sigma_{2k_2}^c \cdots \sigma_{2k_l}^c}{k_1 k_2 \cdots k_l} \right) - \tfrac{(-)^n}{[2n]!} \right\} \end{aligned} \tag{43}$$

Equivalently,

$$\sigma_{2n}^c = n \sum_{l=1}^{n} \frac{(-)^l}{l} \left\{ \sum_{(k_1,k_2,\cdots k_l)} L_{2k_1}^c L_{2k_2}^c \cdots L_{2k_l}^c \right\} \tag{44}$$

where as in Eq.(43)

$$L_{2m}^c \equiv \frac{(-)^m}{[2m]!} \tag{45}$$

For the $\sin_q(z)$ function, we find

$$\begin{aligned} \sin_q(z) &= z \exp\{b^s(z)\} \\ b^s(z) &= \sum_{i=1}^{\infty} \ln\left(1 - \left(\tfrac{z}{s_i}\right)^2\right) = -\sum_{n=1}^{\infty} \tfrac{1}{n} \sigma_{2n+1}^s z^{2n}, |z| < |s_1| \end{aligned} \tag{46}$$

where

$$\sigma_{2n+1}^s \equiv \sum_{i=1}^{\infty} \left(\frac{1}{s_i^2}\right)^n . \tag{47}$$

The values of these sine sum rules are

$$\begin{aligned}
\sigma_3^s &= \sum_{i=1}^{\infty}\left(\frac{1}{s_i}\right)^2 = \frac{1}{[3]!} \\
\sigma_5^s &= \sum_{i=1}^{\infty}\left(\frac{1}{s_i}\right)^4 = \left(\frac{1}{[3]!}\right)^2 - \frac{2}{[5]!} \\
\sigma_7^s &= \sum_{i=1}^{\infty}\left(\frac{1}{s_i}\right)^6 = \left(\frac{1}{[3]!}\right)^3 - \frac{3}{[3]![5]!} + \frac{3}{[7]!} \\
\sigma_{2n+1}^s &= n\left\{\sum_{l=2}^{n}\frac{(-)^l}{l!}\left(\sum_{(k_1,k_2,\cdots k_l)}\frac{\sigma_{2k_1+1}^s\sigma_{2k_2+1}^s\cdots\sigma_{2k_l+1}^s}{k_1k_2\cdots k_l}\right) - \frac{(-)^n}{[2n+1]!}\right\}
\end{aligned} \tag{48}$$

Equivalently,

$$\sigma_{2n+1}^s = n\sum_{l=1}^{n}\frac{(-)^l}{l}\left\{\sum_{(k_1,k_2,\cdots k_l)} L_{2k_1+1}^s L_{2k_2+1}^s \cdots L_{2k_l+1}^s\right\} \tag{49}$$

where as in Eq.(48)

$$L_{2m+1}^s \equiv \frac{(-)^m}{[2m+1]!} \tag{50}$$

5 Concluding Remarks:

(1) The above sum rules and logarithmic results are representation independent; i.e. they also hold for Jackson's q-exponential function $E_q(z)$, its derivatives, integrals, and as well for its associated trigonometic functions $Cos_q(z)$ and $Sin_q(z)$. The only change is that the bracket, or deformed integer, $[n]$ is to be replaced by $[n]_J \equiv \frac{1-q^n}{1-q}$.

Since [7,5] the zeros of $E_q(z)$ for $q > 1$ are at

$$z_i^E = \frac{q^i}{1-q}, \tag{51}$$

simple expressions follow: The values of the associated sum rules are

$$\begin{aligned}
\sigma_n^E &\equiv \textstyle\sum_{i=1}^{\infty}\left(\frac{1}{z_i^E}\right)^n \\
&= -\frac{(1-q)^n}{1-q^n} \\
&= -\frac{(1-q)^{n-1}}{[n]_J}.
\end{aligned} \tag{52}$$

A power series representation for the associated natural logarithm is

$$\begin{aligned} b^E(z) &\equiv \ln\{E_q(z)\} \\ &= \textstyle\sum_{i=1}^{\infty} \frac{(1-q)^n}{n(1-q^n)} z^n \\ &= \textstyle\sum_{i=1}^{\infty} \frac{(1-q)^{n-1}}{n[n]_J} z^n, \; |z| < |\frac{q}{1-q}|. \end{aligned} \tag{53}$$

For both representations, $[n]$ and $[n]_J$, of the derivatives and integrals of $e_q(z)$, and of the $\cos_q(z)$ and $\sin_q(z)$ functions, asymptotic formula for their associated zeros are given in Ref.[5] so simple expressions also follow for their σ_n's and $b(z)$'s in the regions where these asymptotic formula apply.

(2) Useful checks on the above results and for use in applications of them include:

(i) in the bosonic CS(coherent state) limit $q \rightarrow 1$, the normal numerical values must be obtained,

(ii) in the $q \rightarrow 0$ limit, results corresponding [9] to fermionic CS's should be obtained [this is a quick, though quite trivial, check],

(iii) by the use of $[n] \rightarrow [n]_J \equiv \frac{1-q^n}{1-q}$, the known exact zeros of $E_q(z)$ for $q > 1$ can be used for non-trivial checks. These zeros are at $z_i^E = q^n/(1-q)$.

(3) The determination of the series expansion and a general representation for the usual natural logarithm for the q-exponential function, $b(z) = \ln\{e_q(z)\}$, means that the q-analogue coherent states can now be written in the form of an exponential operator acting on the vacuum state:

$$\begin{aligned} |z\rangle_q &= N(|z|) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{[n]!}} |n\rangle_q \\ &= N(|z|) \exp\{b(za^+)\} |0\rangle_q \end{aligned} \tag{54}$$

where

$$
\begin{aligned}
b(za^+) &= \sum_{i=1}^{\infty} \ln\left\{1 - \tfrac{za^+}{z_i}\right\} \\
b(za^+) &= za^+ - \tfrac{1}{[2]!}(za^+)^2 - \left\{\tfrac{1}{[3]!} - 2\left(\tfrac{1}{[2]!}\right)^2\right\}(za^+)^3 \\
&\quad - \left\{\tfrac{1}{[4]!} - \tfrac{5}{[3]![2]!} + 5\left(\tfrac{1}{[2]!}\right)^3\right\}(za^+)^4 + \ldots
\end{aligned} \tag{55}
$$

(4)The successful evaluations and applications of the sum rules for the q-trigonometric functions motivate the following definitions of q-analogue generalizations of the usual Bernoulli numbers:

$$
\begin{aligned}
\tfrac{2^{2n-1}}{(2n)!} B_n^q &\equiv \sum_{i=1}^{\infty} \left(\tfrac{1}{s_i}\right)^{2n} \\
&= \sigma^s_{2n+1}
\end{aligned} \tag{56}
$$

$$
\begin{aligned}
\tfrac{2^{2n-1}}{(2n)!} \widetilde{B}_n^q &\equiv \tfrac{1}{(2^{2n}-1)} \sum_{i=1}^{\infty} \left(\tfrac{1}{c_i}\right)^{2n} \\
&= \tfrac{1}{(2^{2n}-1)} \sigma^c_{2n}
\end{aligned} \tag{57}
$$

Hence, under q-deformation, the usual Bernoulli numbers become the values of the sum rules for the reciprocals of the zeros of the q-analogue trigonometric functions, $cos_q(z)$ and $sin_q(z)$. For the Riemann zeta function, these results do not yield a unique definition. However, analogous simple definitions for p complex are

$$
\frac{1}{\pi^p} \zeta_q(p) \equiv \sum_{i=1}^{\infty} \left(\frac{1}{s_i}\right)^p \tag{58}
$$

$$
\frac{1}{\pi^p} \widetilde{\zeta}_q(p) \equiv \frac{1}{(2^p - 1)} \sum_{i=1}^{\infty} \left(\frac{1}{c_i}\right)^p \tag{59}
$$

"Note added in proof:" The ordinary natural logarithm of $E_q(z)$ for $0 < q < 1$ is shown to be related to a q-analogue dilogarithm, $Li_2(z;q)$, in [10] and in the recent survey of q-special functions by Koornwinder [11]: From Eq.(53) and $E_s(x)E_{1/s}(-x)$ =1, for $0 < q < 1$

$$
\ln\{E_q(\frac{z}{1-q})\} = \sum_{i=1}^{\infty} \frac{1}{n(1-q^n)} z^n \equiv Li_2(z;q) \tag{60}
$$

which is identical with Eq.(53). Formally [10],

$$\lim_{q\uparrow 1}(1-q)Li_2(z;q)=\sum_{n=1}^{\infty}\frac{z^n}{n^2}=Li_2(z) \tag{61}$$

the ordinary Euler dilogarithm. Other recent works on q- exponential functions are in Refs.[12].

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Figure Captions

Figure 1: Plot showing the behaviour of the q-analogue exponential function $e_q(x)$ for x negative. The $q = 0.1$ curve displays the universal behaviour of $e_q(x)$ for $q < q_1^*(q_1^* \approx 0.14)$. As q increases above the first collision point at $q_1^* \approx 0.14$, the zeros, $\mu_i = z_i$, collide in pairs and then move off into the complex z plane. They move off as (and remain) a complex conjugate pair. The $q = 0.2$ curve displays the behaviour of $e_q(x)$ after the collision of the first pair of zeros μ_1, μ_2 but before the collsion of the first pair of turning points. The first two turning points τ_1, τ_2 collide at $q^* \approx 0.25$. The turning points τ_i of $e_q(z)$ are mapped into the branch points b_i, of $ln_q(w)$.

Figure 2: These two figures and Figs. 3a and 3b show the Riemann sheet structure and the mappings of Jackson's exponential function $E_q(z)$ and of its inverse function $Ln_q(w)$ for $q^E = 1.09$. For instance, $w = E_q(z)$ maps the region labeled "$1, 2, 1_L, 2_L$"
in Fig. 2b onto the upper-half-plane (uhp) of the first w sheet for $Ln_q(w)$, see Fig. 3a. The turning points τ_1, τ_2 are mapped respectively into the branch points b_1, b_2 of Fig. 3a. These figures suffice to illustrate the behaviour of $E_q(z)$ and $Ln_q(w)$ for all $q^E > 1$ because as $q^E \to 1$, the zeros and turning points of $E_q(z)$ do not collide, but simply move along the negative x axis and out to infinity. In the complex w plane the associated branch points of $Ln_q(w)$ all move into the origin. This limit thereby gives the usual Riemann surface for $exp(z)$ and $ln(w)$. Figs. 2 and 3 also illustrate the Riemann surface for $e_q(z)$ and $ln_q(w)$ but only prior to the collision of the first pair of zeros, i.e. for $q < q_1^*(q_1^* \approx 0.14)$. Figures 4-8 show the Riemann surfaces of $e_q(z)$ and $ln_q(w)$ for larger q values,$q_1^* < q \leq 1$.

Figure 3: (a) The first upper sheet of $Ln_q(w)$ for $q^E = 1.09$. The turning points τ_1, τ_2 in Fig. 2 for $E_q(z)$ are mapped respectively into the square-root branch points b_1, b_2 of Fig. 3a, 3b for

$Ln_q(w)$. An "opening spiral", instead of the usual unit circle, is the "image" of the positive y axis (the $x = 0$ line) in Fig. 2. The first lower sheet of $Ln_q(w)$ is the mirror image of this figure (the reflection is thru the horizontal u axis); the lower sheets corresponding to the other "upper sheet" figures in this paper are similarly obtained. (b) The second upper sheet of $Ln_q(w)$ for $q^E = 1.09$. Note that the opening spiral continues that in (a). The cut above the real axis from b_2 to ∞ goes back down to the first sheet, Fig. 3a.

Figure 4: This figure and Fig. 5 show the Riemann sheet structure and the mappings of $e_q(z)$ and of its inverse function $ln_q(w)$ for $0.14 < q \approx 0.22 < 0.25$. For this range of q, the first two zeros μ_1, μ_2 of $e_q(x)$ have collided and have moved off as a complex conjugate pair $\mu_A, \mu_{\bar{A}}$; the μ_A zero is marked in this figure. Note that as in Fig. 2, $Im\{e_q(z)\} = 0$ on all "solid" contour lines, whereas $Re\{e_q(z)\} = 0$ on all "dashed" coutour lines.

Figure 5: The first upper sheet for $ln_q(w)$ for $0.14 < q \approx 0.22 < 0.25$. When q is increased to $q \approx 0.25$, the branch points $b_1 = b_2$ coincide since the turning points τ_1, τ_2 of Fig. 4 have collided. Then, the branch cut to the first lower sheet nolonger exists. τ_1, τ_2 become a complex conjugate pair $\tau_A, \tau_{\bar{A}}$ and move off into the complex z plane, as shown in Figs. 6-8.

Figure 6: This figure and Figs. 7-8 show the Riemann sheet structure and the mappings of $e_q(z)$ and of its inverse function $ln_q(w)$ for $q \approx 0.35$. The first two turning points τ_1, τ_2 of $e_q(x)$ have collided and have moved off as a complex conjugate pair $\tau_A, \tau_{\bar{A}}$; the τ_A turning point is marked in this figure, $\tau_A = -3.54 + i1.33$. The line corresponding to the $\alpha' \beta'$ branch cut thru b_A, see Figs. 7-8, is the wiggly line from α on the $x < 0$ axis, thru τ_A, and on to β on the $Im\{e_q(z)\} = 0$ curve. τ_A(and b_A) are fixed, but α and β(α' and β') are simple though arbitrary positions on their respective $Im\{e_q(z)\} = 0$ lines. The third zero μ_3 of $e_q(z)$ is still on the $x < 0$ axis.

Figure 7: (a) The first upper sheet of $ln_q(w)$ for $q = 0.35$. The image of the $x = 0$ line in the complex z plane is shown. (b) An enlargement of the first quadrant which shows the $\alpha' \beta'$ branch cut. For clarity of illustration, the position of b_A has been displaced from its true position at $b_A = 0.0222 + i0.0188$.

Figure 8: The second upper sheet of $ln_q(w)$ for $q = 0.35$. The b_A square-root branch point only occurs on the first two upper sheets, i.e. in Fig. 7 and here. The α' point (not shown) lies opposite the β' point and to the left of the b_A cut structure.

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cnelson@bingvmb.cc.binghamton.edu